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# ON THE ILL-POSEDNESS OF THE CUBIC NONLINEAR SCHRÖDINGER EQUATION ON THE CIRCLE

TADAHIRO OH AND YUZHAO WANG

**ABSTRACT.** In this note, we consider the ill-posedness issue for the cubic nonlinear Schrödinger equation (NLS) on the circle. In particular, adapting the argument by Christ-Colliander-Tao [14] to the periodic setting, we exhibit a norm inflation phenomenon for both the usual cubic NLS and the Wick ordered cubic NLS for  $s \leq s_{\text{crit}} := -\frac{1}{2}$ . We also discuss norm inflation phenomena for general cubic fractional NLS on the circle.

## 1. INTRODUCTION

We consider the cubic nonlinear Schrödinger equation (NLS):

$$\begin{cases} i\partial_t u - \partial_x^2 u \pm |u|^2 u = 0 \\ u|_{t=0} = u_0, \end{cases} \quad (x, t) \in \mathbb{T} \times \mathbb{R} \quad \text{or} \quad \mathbb{R} \times \mathbb{R}. \quad (1.1)$$

The equation (1.1) appears in various physical settings: nonlinear optics, fluids, plasmas, and quantum field theory. See [35] for a general review on the subject. It is also known to be one of the simplest completely integrable partial differential equations (PDEs) [1, 2, 19].

**1.1. Galilean invariance.** The Cauchy problem for the cubic NLS has been studied extensively both on  $\mathbb{R}$  and  $\mathbb{T}$ . It is well known that (1.1) is invariant under several symmetries. The Galilean invariance states that if  $u(x, t)$  is a solution to (1.1) on  $\mathbb{R}$ , then  $u^\beta(x, t) := e^{i\frac{\beta}{2}x} e^{i\frac{\beta^2}{4}t} u(x + \beta t, t)$  is also a solution to (1.1) on  $\mathbb{R}$  with modulated initial data. Note that the  $L^2$ -norm is preserved under this Galilean invariance.<sup>1</sup> Namely,  $s_{\text{crit}}^\infty := 0$  is the critical Sobolev regularity with respect to the Galilean invariance. In fact, while (1.1) is globally well-posed in  $L^2(\mathbb{R})$  [36], it is known to be ‘mildly ill-posed’ below  $L^2(\mathbb{R})$ . More precisely, the solution map  $\Phi(t) : u_0 \in H^s(\mathbb{R}) \mapsto u(t) \in H^s(\mathbb{R})$  fails to be locally uniformly continuous if  $s < 0$  [25, 12]. We point out that this mild ill-posedness result does not assert that (1.1) on  $\mathbb{R}$  is ill-posed below  $L^2(\mathbb{R})$ . Indeed, the well-posedness issue of (1.1) on  $\mathbb{R}$  below  $L^2(\mathbb{R})$  is a long-standing open problem.

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<sup>1</sup>In fact, any Fourier-Lebesgue  $\mathcal{FL}^p$ -norm defined in (1.4) is invariant under the Galilean symmetry.

The Galilean invariance also holds in the periodic setting for  $\beta \in 2\mathbb{Z}$ . In this case, there is even a stronger dichotomy in the behavior of solutions for  $s \geq 0$  and  $s < 0$ . On the one hand, Bourgain [5] proved global well-posedness of (1.1) in  $L^2(\mathbb{T})$ . On the other hand, (1.1) on  $\mathbb{T}$  is known to be ill-posed below  $L^2(\mathbb{T})$ . As in the real line case, the solution map  $\Phi(t) : u_0 \in H^s(\mathbb{T}) \mapsto u(t) \in H^s(\mathbb{T})$  fails to be locally uniformly continuous if  $s < 0$  [7, 12]. Moreover, Christ-Colliander-Tao [13] and Molinet [28] proved that the solution map is in fact discontinuous if  $s < 0$ . Hence, (1.1) on  $\mathbb{T}$  is ill-posed in negative Sobolev spaces. Finally, Guo-Oh [22] proved non-existence of solutions for (1.1) on  $\mathbb{T}$  below  $L^2(\mathbb{T})$ . See [22] for a precise statement.

Therefore, in the periodic setting, one needs to consider a renormalized cubic NLS outside  $L^2(\mathbb{T})$ . Given a global solution  $u \in C(\mathbb{R}; L^2(\mathbb{T}))$  to (1.1), we define the following invertible gauge transformation:

$$u(t) \mapsto \mathcal{G}(u)(t) := e^{\mp 2it \int |u|^2 dx} u(t). \quad (1.2)$$

In view of the  $L^2$ -conservation, a direct computation shows that the gauged function, which we still denote by  $u$ , solves the following *Wick ordered cubic NLS*:

$$\begin{cases} i\partial_t u - \partial_x^2 u \pm (|u|^2 - 2 \int |u|^2 dx) u = 0 \\ u|_{t=0} = u_0, \end{cases} \quad (x, t) \in \mathbb{T} \times \mathbb{R}. \quad (1.3)$$

Conversely, given a global solution  $u \in C(\mathbb{R}; L^2(\mathbb{T}))$  to (1.3), we see that  $\mathcal{G}^{-1}(u)$  solves the original cubic NLS (1.1). Such a gauge transformation, however, does not make sense below  $L^2(\mathbb{T})$  and thus we cannot freely convert solutions of (1.3) into solutions of (1.1). In other words, the standard cubic NLS (1.1) and the Wick ordered cubic NLS (1.3) represent the same dynamics under different gauge choices (only) in  $L^2(\mathbb{T})$ .

The renormalization of the nonlinearity in (1.3) is canonical, corresponding to the Wick ordering in Euclidean quantum field theory. See [6, 30, 31]. The specific choice of a gauge for (1.3) removes a certain singular component from the cubic nonlinearity. As a result, the Wick ordered cubic NLS (1.3) is known to behave better than the cubic NLS (1.1) outside  $L^2(\mathbb{T})$ . In fact, the standard cubic NLS on  $\mathbb{R}$  and the Wick ordered cubic NLS (1.3) on  $\mathbb{T}$  share many common features:

- global well-posedness in  $L^2$  [36, 5].
- failure of uniform continuity of the solution map on bounded sets below  $L^2$  [25, 7, 12, 16].
- weak continuity in  $L^2$  [18, 30].
- local well-posedness in the Fourier-Lebesgue spaces  $\mathcal{FL}^{s,p}$  [20, 11, 21] for  $s \geq 0$  and  $1 < p < \infty$ . Here, the Fourier-Lebesgue space  $\mathcal{FL}^{s,p}(\mathbb{R})$  is defined by the norm:

$$\|f\|_{\mathcal{FL}^{s,p}(\mathbb{R})} = \|\langle \xi \rangle^s \widehat{f}(\xi)\|_{L^p(\mathbb{R})} \quad (1.4)$$

with an analogous definition for  $\mathcal{FL}^{s,p}(\mathbb{T})$  in the periodic setting. When  $s = 0$ , we set  $\mathcal{FL}^p := \mathcal{FL}^{0,p}$  for simplicity.

- a priori bound and existence (without uniqueness) of solutions in negative Sobolev spaces [26, 15, 27, 22].

See [30] for more discussion on this issue.

**1.2. Scaling symmetry.** Another important symmetry is the dilation symmetry; if  $u(x, t)$  is a solution to (1.1) on  $\mathbb{R}$ , then  $u^\lambda(x, t) := \lambda^{-1}u(\lambda^{-1}x, \lambda^{-2}t)$  is also a solution to (1.1) on  $\mathbb{R}$  with scaled initial data. This dilation symmetry induces another critical Sobolev regularity given by  $s_{\text{crit}} := -\frac{1}{2}$ . Namely, the dilation symmetry leaves the homogeneous  $\dot{H}^{s_{\text{crit}}}$ -norm invariant. It is commonly conjectured that a PDE is ill-posed in  $H^s$  for  $s < s_{\text{crit}}$ . Indeed, Christ-Colliander-Tao [14] exhibited a *norm inflation* phenomenon for (1.1) in  $H^s(\mathbb{R})$  for  $s < s_{\text{crit}} = -\frac{1}{2}$ .<sup>2</sup> Note that this is a stronger statement than the failure of continuity of the solution map (at 0). On the one hand, the failure of continuity at 0 states that given any  $\varepsilon > 0$ , there exist a solution  $u = u(\varepsilon)$  to (1.1) and a time  $t \in (0, \varepsilon)$  such that

$$\|u(0)\|_{H^s(\mathbb{R})} < \varepsilon \quad \text{and} \quad \|u(t)\|_{H^s(\mathbb{R})} \gtrsim 1.$$

On the other hand, norm inflation states that given any  $\varepsilon > 0$ , there exist a solution  $u = u(\varepsilon)$  to (1.1) on  $\mathbb{R}$  and a time  $t \in (0, \varepsilon)$  such that

$$\|u(0)\|_{H^s(\mathbb{R})} < \varepsilon \quad \text{and} \quad \|u(t)\|_{H^s(\mathbb{R})} > \varepsilon^{-1}.$$

While there is no dilation symmetry in the periodic setting, the heuristics provided by the scaling argument still plays an important role. In the following, we prove that the same norm inflation phenomenon also holds for both the standard cubic NLS (1.1) and the Wick ordered NLS (1.3) in the periodic setting if  $s \leq -\frac{1}{2}$ .<sup>3</sup>

**Theorem 1.1.** *Let  $s \leq -\frac{1}{2}$ . Then, given any  $\varepsilon > 0$ , there exist a solution  $u = u(\varepsilon)$  to (1.1) on  $\mathbb{T}$  and a time  $t \in (0, \varepsilon)$  such that*

$$\|u(0)\|_{H^s(\mathbb{T})} < \varepsilon \quad \text{and} \quad \|u(t)\|_{H^s(\mathbb{T})} > \varepsilon^{-1}.$$

While we state and prove Theorem 1.1 for the standard cubic NLS (1.1) on  $\mathbb{T}$ , it is easy to see that the same result holds for the Wick ordered NLS

<sup>2</sup> Although Theorem 2 in [14] claims norm inflation for  $s \leq -\frac{d}{2}$ , the proof requires  $s < s_{\text{crit}}$  and thus, as it is written, their result does not hold for the cubic NLS (1.1) on  $\mathbb{R}$  at the scaling critical regularity  $s = -\frac{1}{2}$ . See (4.4) in [14]. One can, however, modify the argument and show that norm inflation holds even at the scaling critical regularity. See Subsection 3.3 below. See also [24, 29, 10].

<sup>3</sup>In a recent paper [29], the first author showed that the norm inflation for (1.1) on  $\mathcal{M} = \mathbb{R}$  or  $\mathbb{T}$  also holds for general initial data in the following sense; let  $s \leq -\frac{1}{2}$ . Then, given any  $u_0 \in H^s(\mathcal{M})$  and any  $\varepsilon > 0$ , there exist a solution  $u = u(\varepsilon, u_0)$  to (1.1) on  $\mathcal{M}$  and a time  $t \in (0, \varepsilon)$  such that

$$\|u(0) - u_0\|_{H^s(\mathcal{M})} < \varepsilon \quad \text{and} \quad \|u_\varepsilon(t)\|_{H^s(\mathcal{M})} > \varepsilon^{-1}.$$

This in particular implies that the solution map to the cubic NLS (1.1) is discontinuous everywhere in  $H^s(\mathcal{M})$ . A similar norm inflation with general initial data holds for the Wick ordered NLS (1.3) on  $\mathbb{T}$ .

(1.3) on  $\mathbb{T}$  since the gauge transformation (1.2) preserves any Sobolev norm (thanks to the  $L^2$ -conservation).

We prove Theorem 1.1 by adapting the argument in [14] to the periodic setting. In particular, we exploit the high-to-low energy transfer in the dynamics of the following (dispersionless) ODE:

$$i\partial_t w \pm |w|^2 w = 0 \quad (1.5)$$

and the small dispersion NLS:

$$i\partial_t v - \delta^2 \partial_x^2 v \pm |v|^2 v = 0,$$

approximating (1.5). In [14], scaling plays an important role. In the periodic setting, however, scaling alters spatial domains and thus we must work with care. We also use the periodization by Dirac's comb to transfer some information from  $\mathbb{R}$  to dilated tori of large periods. We point out that in the critical case:  $s = -\frac{1}{2}$ , we employ solutions to (1.5) exhibiting stronger high-to-low energy transfer than those in [14]. Moreover, we do not use a scaling argument in this case. Instead, we directly approximate NLS (1.1) by the ODE (1.5). See Subsection 3.3 and Appendix B. While this argument can be adapted to the higher dimensional setting (on both rational and irrational tori), we do not pursue this issue here.

In [23], Iwabuchi-Ogawa introduced a technique for proving ill-posedness of evolution equations, exploiting high-to-low energy transfer in the first Picard iterate. This method is built upon the previous work by Bejenaru-Tao [3] and is developed further to cope with a wider class of equations, utilizing (scaled) modulation spaces. In fact, Theorem 1.1 can also be proven by this method. See Kishimoto [24] for details. See also a recent result by Carles-Kappeler [9], where they employed a geometric optics approach and proved a norm inflation for (1.1) and (1.3) in the Fourier-Lebesgue space  $\mathcal{FL}^{s,p}(\mathbb{T})$  for  $s < -\frac{2}{3}$  and  $p \in [1, \infty]$ . While our method in this paper can be adapted to the Fourier-Lebesgue spaces, we do not pursue this issue here.

Lastly, we point out the work by Burq-Gérard-Tzvetkov [8, Appendix] on ill-posedness in  $H^1(\mathcal{M})$  of super-quintic NLS on a three-dimensional Riemannian manifold  $\mathcal{M}$ . While this work is strongly motivated by [14] utilizing the dispersionless NLS, they exploit the local-in-space nature of the dynamics responsible for norm inflation in the absence of dilation symmetry. This would provide an alternative approach to Theorem 1.1.

**1.3. Norm inflation for the cubic fractional NLS.** In the following, we briefly discuss the situation for the cubic fractional NLS on the circle:

$$\begin{cases} i\partial_t u + (-\partial_x^2)^\alpha u \pm |u|^2 u = 0 \\ u|_{t=0} = u_0, \end{cases} \quad (x, t) \in \mathbb{T} \times \mathbb{R}, \quad (1.6)$$

for  $\alpha > 0$ . When  $\alpha = \frac{1}{2}$ , (1.6) corresponds to the cubic half-wave equation, while it corresponds to the cubic fourth order NLS when  $\alpha = 2$ . See, for example, [17, 32, 33] for the work on (1.6) in these cases.

The dilation symmetry for (1.6) on  $\mathbb{R}$  states that if  $u(x, t)$  is a solution to (1.6) on  $\mathbb{R}$ , then  $u^\lambda(x, t) := \lambda^{-\alpha} u(\lambda^{-1}x, \lambda^{-2\alpha}t)$  is also a solution to (1.6) on  $\mathbb{R}$  with scaled initial data. This gives rise to the scaling critical Sobolev regularity  $s_{\text{crit}}^\alpha := \frac{1}{2} - \alpha$ .

We prove the following norm inflation phenomenon for (1.6).

**Theorem 1.2.** *Given  $\alpha > 0$ , let (i)  $s \leq s_{\text{crit}}^\alpha$  if  $s_{\text{crit}}^\alpha \leq -\frac{1}{2}$  or (ii)  $s < s_{\text{crit}}^\alpha$  and  $s \neq 0$  if  $s_{\text{crit}}^\alpha > -\frac{1}{2}$ . Then, given any  $\varepsilon > 0$ , there exist a solution  $u = u(\varepsilon)$  to (1.6) on  $\mathbb{T}$  and a time  $t \in (0, \varepsilon)$  such that*

$$\|u(0)\|_{H^s(\mathbb{T})} < \varepsilon \quad \text{and} \quad \|u(t)\|_{H^s(\mathbb{T})} > \varepsilon^{-1}.$$

Note that, even if  $s_{\text{crit}}^\alpha \geq 0$ , there is no norm inflation for  $s = 0$  due to the  $L^2$ -conservation. When  $s \leq -\frac{1}{2}$ , Theorem 1.1 follows from a small modification of the proof of Theorem 1.1. When  $s > -\frac{1}{2}$ , we need to use different energy transfer mechanisms, depending on  $-\frac{1}{2} < s < 0$  or  $s > 0$ . While we state and prove Theorem 1.2 in the periodic setting, the proof of Theorem 1.2 can be easily adapted to the non-periodic setting (and it is in fact easier).

In a recent paper [10], Choffrut-Pocovnicu applied the argument in [23, 24] and proved norm inflation for the cubic half-wave equation on  $\mathbb{T}$  and  $\mathbb{R}$  ((1.6) with  $\alpha = \frac{1}{2}$ ) for  $s < s_{\text{crit}}^\alpha = 0$ , exploiting abundance of resonances in the half-wave equation. They also obtained a partial result for general  $\alpha > 0$ .

**Remark 1.3.** When  $0 < \alpha < 1$ , we have  $s_{\text{crit}}^\alpha > -\frac{1}{2}$ . In this case, the question of norm inflation at the critical regularity  $s = s_{\text{crit}}^\alpha > -\frac{1}{2}$  with  $s \neq 0$  remains open. See Remarks 4.5 and B.5 for a brief discussion for  $-\frac{1}{2} < s_{\text{crit}}^\alpha < 0$ .

The defocusing/focusing nature of the equation does not play any role. Hence, we assume that it is defocusing (with the  $+$  sign in (1.1) and (1.6)) in the following. Moreover, in view of the time reversibility of the equations, we only consider positive times.

## 2. SOBOLEV SPACES ON A SCALED TORUS AND DIRAC'S COMB

In the proof of Theorem 1.1, scaling of periodic domains plays an important role. Hence, we briefly go over the basic definitions and properties of Sobolev spaces on a scaled torus  $\mathbb{T}_L := \mathbb{R}/(L\mathbb{Z})$ ,  $L \geq 1$ .<sup>4</sup> Given a function  $f$  on  $\mathbb{T}_L$ , we define its Fourier coefficient by

$$\widehat{f}\left(\frac{n}{L}\right) = \frac{1}{L} \int_{\mathbb{T}_L} f(x) e^{-2\pi i \frac{n}{L}x} dx, \quad n \in \mathbb{Z}. \quad (2.1)$$

Then, we have the following Fourier inversion formula:

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}\left(\frac{n}{L}\right) e^{2\pi i \frac{n}{L}x} \quad (2.2)$$

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<sup>4</sup>In the following, we often identify  $\mathbb{T}_L$  with the interval  $[-\frac{L}{2}, \frac{L}{2}] \subset \mathbb{R}$ .

and Plancherel's identity:

$$\|f\|_{L^2(\mathbb{T}_L)} = L^{\frac{1}{2}} \|\widehat{f}\|_{\ell^2(\mathbb{Z}/L)} = L^{\frac{1}{2}} \left( \sum_{n \in \mathbb{Z}} |\widehat{f}(\frac{n}{L})|^2 \right)^{\frac{1}{2}}.$$

Given  $s \in \mathbb{R}$ , we define the homogeneous Sobolev space  $\dot{H}^s(\mathbb{T}_L)$  and the inhomogeneous Sobolev space  $H^s(\mathbb{T}_L)$  by the norms:

$$\|f\|_{\dot{H}^s(\mathbb{T}_L)} := L^{\frac{1}{2}} \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} |\frac{n}{L}|^{2s} |\widehat{f}(\frac{n}{L})|^2 \right)^{\frac{1}{2}}, \quad (2.3)$$

$$\|f\|_{H^s(\mathbb{T}_L)} := L^{\frac{1}{2}} \left( \sum_{n \in \mathbb{Z}} \langle \frac{n}{L} \rangle^{2s} |\widehat{f}(\frac{n}{L})|^2 \right)^{\frac{1}{2}}, \quad (2.4)$$

where  $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ .

Recall the following definition of Dirac's comb:

$$\Xi_L(x) = \sum_{n \in \mathbb{Z}} \delta_0(x - nL)$$

for  $L > 0$ , where  $\delta_0$  is the Dirac's delta function. Given a (smooth) function  $f \in L^1(\mathbb{R})$ , define the periodization  $f_L$  of  $f$  with period  $L$  by setting

$$f_L(x) = \Xi_L * f(x) = \sum_{n \in \mathbb{Z}} f(x - nL). \quad (2.5)$$

Then,  $f_L$  is a periodic function with period  $L$ . Moreover, from (2.1) and (2.5), we have

$$\widehat{f_L}(\frac{n}{L}) = \frac{1}{L} \widehat{f}(\frac{n}{L}), \quad n \in \mathbb{Z}, \quad (2.6)$$

where  $\widehat{f}$  on the right-hand side denotes the Fourier transform of  $f$  on  $\mathbb{R}$ . Then, it follows from (2.3) and (2.6) with a Riemann sum approximation that

$$\begin{aligned} \|f_L\|_{\dot{H}^s(\mathbb{T}_L)} &= \left( \frac{1}{L} \sum_{n \in \mathbb{Z} \setminus \{0\}} |\frac{n}{L}|^{2s} |\widehat{f}(\frac{n}{L})|^2 \right)^{\frac{1}{2}} \\ &\longrightarrow \left( \int_{\mathbb{R}} |\xi|^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} = \|f\|_{\dot{H}^s(\mathbb{R})} \end{aligned} \quad (2.7)$$

as  $L \rightarrow \infty$ . Similarly, we have

$$\|f_L\|_{H^s(\mathbb{T}_L)} \longrightarrow \|f\|_{H^s(\mathbb{R})} \quad (2.8)$$

as  $L \rightarrow \infty$ . Dirac's comb and related Poisson's summation formula are typical tools to pass information from  $\mathbb{R}$  to a periodic torus. See [4] for example.

## 3. PROOF OF THEOREM 1.1

We follow closely the norm inflation argument for (1.1) on  $\mathbb{R}$  by Christ-Colliander-Tao [14] and adapt it suitably to the periodic setting, at least when  $s < -\frac{1}{2}$ . Recall that the argument in [14] consists of the following three components:

- (a) high-to-low energy transfer for solutions to the following ODE:

$$i\partial_t w(x) + |w|^2 w(x) = 0 \quad (3.1)$$

for  $x \in \mathbb{R}$ . Recall that there is an explicit solution formula for (3.1):

$$w(x, t) = w(x, 0)e^{i|w(x, 0)|^2 t}. \quad (3.2)$$

- (b) approximation property of the ODE (3.1) by the following cubic NLS with small dispersion:

$$i\partial_t v - \delta^2 \partial_x^2 v + |v|^2 v = 0. \quad (3.3)$$

- (c) modified scaling argument to relate the small dispersion NLS (3.3) and the cubic NLS (1.1).

A primary issue in adapting this argument to the periodic setting is the fact that the dilation symmetry alters the spatial domain in the periodic setting. In the following, we use Dirac's comb and periodization (of a function on  $\mathbb{R}$ ) with large periods to handle Step (b). Then, we apply a modified scaling to reduce (3.3) on a torus of a large period to (1.1) on the standard torus  $\mathbb{T}$ .

In the critical case  $s = -\frac{1}{2}$ , the scaling part of the argument no longer works. We directly establish an approximation property of the ODE (3.1) by NLS (1.1).

**3.1. Approximation lemma.** In the following, we establish an approximation property of the ODE (3.1) by the small dispersion NLS (3.3). This approximation lemma will be useful in the supercritical case discussed in Subsection 3.2. See [14, Lemma 2.1] for an analogous statement in the non-periodic setting.

**Lemma 3.1.** *Given  $\phi \in C_c^\infty(\mathbb{R})$ , suppose that  $L$  is sufficiently large such that  $\text{supp } \phi \subset [-\frac{L}{2}, \frac{L}{2}] \simeq \mathbb{T}_L$ . Let  $w$  and  $v = v(\delta)$  be the solutions to (3.1) and (3.3) on  $\mathbb{T}_L$  with  $w|_{t=0} = v|_{t=0} = \phi$ , respectively. Then, given any  $\beta \in (0, 2)$ , there exist  $c > 0$ ,  $C > 0$ ,  $\delta_0 > 0$ , and  $L_0 \geq 1$  such that*

$$\|v(t) - w(t)\|_{H^1(\mathbb{T}_L)} \leq C\delta^\beta$$

for all  $|t| \leq c|\log \delta|^c$ , all  $0 < \delta \leq \delta_0$ , and all periods  $L \geq L_0$ .

In the current periodic setting, it is important to prove Lemma 3.1 with constants  $c, C$ , and  $\delta_0$ , independent of a period  $L \geq 1$ . For this purpose, we recall the following Sobolev's embedding type estimate. From (2.2), (2.4),



and a Riemann sum approximation, we have

$$\begin{aligned} \|f\|_{L^\infty(\mathbb{T}_L)} &\leq \sum_{n \in \mathbb{Z}} |\widehat{f}(\frac{n}{L})| \leq \left( \sum_{n \in \mathbb{Z}} \frac{1}{1 + |\frac{n}{L}|^2} \cdot \frac{1}{L} \right)^{\frac{1}{2}} \|f\|_{H^1(\mathbb{T}_L)} \\ &\leq C \|f\|_{H^1(\mathbb{T}_L)}, \end{aligned} \quad (3.4)$$

where  $C > 0$  is independent of  $L \gg 1$ .

*Proof.* Letting  $z := v - w$ , we have

$$\begin{cases} i\partial_t z - \delta^2 \partial_x^2 z = \delta^2 \partial_x^2 w - \mathcal{N}(w, z) \\ z|_{t=0} = 0, \end{cases} \quad (3.5)$$

for  $(x, t) \in \mathbb{T}_L \times \mathbb{R}$ , where  $\mathcal{N}(w, z) := |w + z|^2(w + z) - |w|^2 w$ . Assuming the following bootstrap hypothesis:

$$\|z(t)\|_{H^1(\mathbb{T}_L)} \lesssim 1, \quad (3.6)$$

we need to prove

$$\|z(t)\|_{H^1(\mathbb{T}_L)} \leq C\delta^\beta, \quad (3.7)$$

provided

$$|t| \leq c |\log \delta|^c. \quad (3.8)$$

for some small  $c > 0$  and  $0 < \delta \leq \delta_0 \ll 1$ .

It follows from the continuity of  $z(t)$  with  $z(0) = 0$  that there exists  $\varepsilon > 0$  such that (3.6) holds on  $[-\varepsilon, \varepsilon]$ . In the following, we estimate  $\partial_t \|z(t)\|_{H^1(\mathbb{T}_L)}$  on  $[-\varepsilon, \varepsilon]$ .

Let  $L$  be sufficiently large such that  $\text{supp } \phi \subset \mathbb{T}_L$ . Then, noting that  $\text{supp } w(t) = \text{supp } \phi \subset \mathbb{T}_L$  for all  $t \in \mathbb{R}$ , it follows from (3.2) that

$$\|w(t)\|_{H^k(\mathbb{T}_L)} = \|w(t)\|_{H^k(\mathbb{R})} \leq C(1 + |t|)^k \quad (3.9)$$

for  $k \in \mathbb{N}$  and all sufficiently large  $L \gg 1$ . Thus, we have

$$\delta^2 \|\partial_x^2 w(t)\|_{H^1(\mathbb{T}_L)} \leq \delta^2 \|\partial_x^2 w(t)\|_{H^1(\mathbb{R})} \leq C\delta^2(1 + |t|)^3. \quad (3.10)$$

By the product rule, Hölder's inequality, and Sobolev's embedding (3.4) with (3.9), we have

$$\|\mathcal{N}(w, z)(t)\|_{H^1(\mathbb{T}_L)} \leq C(1 + |t|)^2 \|z(t)\|_{H^1(\mathbb{T}_L)} + C\|z(t)\|_{H^1(\mathbb{T}_L)}^3. \quad (3.11)$$

From (3.5), (3.10), and (3.11) with the bootstrap assumption (3.6), we have

$$\partial_t \|z(t)\|_{H^1(\mathbb{T}_L)} \leq C\delta^2(1 + |t|)^3 + C(1 + |t|)^3 \|z(t)\|_{H^1(\mathbb{T}_L)}.$$

Hence, by Gronwall's inequality with  $z(0) = 0$ , we obtain

$$\|z(t)\|_{H^1(\mathbb{T}_L)} \leq C\delta^2 e^{C(1+|t|)^4},$$

which in turn implies (3.7) as long as  $t \in [-\varepsilon, \varepsilon]$  and (3.8) holds. Therefore, by the continuity argument, we conclude that (3.6) and hence (3.7) hold as long as (3.8) holds.  $\square$

**3.2. Supercritical case.** Let  $s < -\frac{1}{2}$ . Our goal is to construct a sequence of solutions  $u_j$  to (1.1),  $j \in \mathbb{N}$ , such that

$$\|u_j(0)\|_{\dot{H}^s(\mathbb{T})} < \frac{1}{j} \quad \text{and} \quad \|u_j(t_j)\|_{\dot{H}^s(\mathbb{T})} > j \quad (3.12)$$

for some  $0 < t_j < \frac{1}{j}$ .

Given  $s < -\frac{1}{2}$ , let  $\phi \in C_c^\infty(\mathbb{R})$  with  $\text{supp } \phi \subset [-K, K]$  for some  $K > 0$ . Moreover, assume that  $\widehat{\phi}(\xi) = O_{\xi \rightarrow 0}(|\xi|^\kappa)$  for some  $\kappa > -s - \frac{1}{2}$  such that  $\phi \in \dot{H}^s(\mathbb{R})$ . Let  $w$  be the solution (3.1) on  $\mathbb{R}$  with  $w|_{t=0} = \phi$ . Note that  $\text{supp } w(t) = \text{supp } \phi \subset [-K, K]$  for all  $t \in \mathbb{R}$ . In view of (3.2), we can choose  $\phi$  such that

$$\left| \int_{\mathbb{R}} w(x, 1) dx \right| \gtrsim 1. \quad (3.13)$$

See Appendix A for details of the construction of such a function  $\phi$ .

Given  $j \in \mathbb{N}$ , let

$$L_j = \frac{\delta_j}{\lambda_j} \quad (3.14)$$

for some  $0 < \lambda_j \ll \delta_j \ll 1$  (to be chosen later). Let  $w_j = \Xi_{L_j} * w$  be the periodization of  $w$  with period  $L_j$  defined in (2.5). By assuming  $L_j \geq 2K$ , we have  $w_j(x) = w(x)$  on  $\mathbb{T}_{L_j} \simeq [-\frac{L_j}{2}, \frac{L_j}{2}] \supset [-K, K]$ . In particular,  $w_j$  is a solution to (3.1) on  $\mathbb{T}_{L_j}$  with  $w_j|_{t=0} = \phi$ .

Let  $v_j$  be the solution to (3.3) with  $\delta = \delta_j$  on  $\mathbb{T}_{L_j}$  such that  $v_j|_{t=0} = \phi$ . By choosing  $\delta_j \leq \delta_0$ , Lemma 3.1 with  $\beta = 1$  yields

$$\|v_j(t) - w_j(t)\|_{H^1(\mathbb{T}_{L_j})} \leq C\delta_j \quad (3.15)$$

for all  $|t| \leq c|\log \delta_j|^c$ . Now, define  $u_j$  by

$$u_j(x, t) = \frac{1}{\lambda_j} v_j\left(\frac{\delta_j x}{\lambda_j}, \frac{t}{\lambda_j^2}\right). \quad (3.16)$$

Then,  $u_j$  is a solution to (1.1) on  $\mathbb{T}$ . From (2.1) with (3.14), we have

$$\begin{aligned} \widehat{u}_j(n, \lambda_j^2 t) &= \int_{\mathbb{T}} u_j(x, \lambda_j^2 t) e^{-2\pi i n x} dx = \frac{1}{\lambda_j} \frac{1}{L_j} \int_{\mathbb{T}_{L_j}} v_j(x, t) e^{-2\pi i \frac{n}{L_j} x} dx \\ &= \frac{1}{\lambda_j} \widehat{v}_j\left(\frac{n}{L_j}, t\right). \end{aligned} \quad (3.17)$$

In particular, from (3.17) and (2.6), we have

$$\widehat{u}_j(0, 0) = \frac{1}{\lambda_j} \widehat{v}_j(0, 0) = \frac{1}{\lambda_j L_j} \mathcal{F}_{\mathbb{R}}(\phi)(0) = 0. \quad (3.18)$$

From (2.3) and (3.17), we have

$$\|u_j(t)\|_{\dot{H}^s(\mathbb{T})} = \lambda_j^{-s-\frac{1}{2}} \delta_j^{s-\frac{1}{2}} \left\| v_j\left(\frac{t}{\lambda_j^2}\right) \right\|_{\dot{H}^s(\mathbb{T}_{L_j})}. \quad (3.19)$$

In view of (3.18), (3.19), and (2.7) with  $v_j(0) = \phi$ , we can choose  $0 < \lambda_j \ll \delta_j \ll 1$  such that

$$\begin{aligned} \|u_j(0)\|_{H^s(\mathbb{T})} &\leq \|u_j(0)\|_{\dot{H}^s(\mathbb{T})} = \lambda_j^{-s-\frac{1}{2}} \delta_j^{s-\frac{1}{2}} \|v_j(0)\|_{\dot{H}^s(\mathbb{T}_{L_j})} \\ &\sim \lambda_j^{-s-\frac{1}{2}} \delta_j^{s-\frac{1}{2}} \|\phi\|_{\dot{H}^s(\mathbb{R})} \ll \frac{1}{j}, \end{aligned} \quad (3.20)$$

provided  $s < -\frac{1}{2}$ . In particular, we can simply choose  $0 < \lambda_j \ll \delta_j \ll 1$  such that

$$\lambda_j^{-s-\frac{1}{2}} \delta_j^{s-\frac{1}{2}} \sim \delta_j^\theta \ll \frac{1}{j} \quad (3.21)$$

for some small  $\theta > 0$  with  $s < -\frac{1}{2} - \frac{\theta}{2}$ .

On the other hand, from (3.13) and the continuity of  $\widehat{w}(\cdot, 1)$ , there exists  $C_0 > 0$  such that

$$|\widehat{w}(\xi, 1)| \geq \frac{c}{2}$$

for all  $|\xi| \leq C_0$ . Then, from (2.6), we have

$$|\widehat{w}_j(\frac{n}{L_j}, 1)| \geq \frac{c}{2L_j} \quad (3.22)$$

for all  $|n| \leq C_0 L_j$ . Define  $\Gamma_j \subset \mathbb{Z}$  by

$$\Gamma_j := \left\{ n \in \mathbb{Z} : |n| \leq C_0 L_j, \left| \widehat{w}_j(\frac{n}{L_j}, 1) - \widehat{v}_j(\frac{n}{L_j}, 1) \right| \geq \frac{c}{4L_j} \right\}. \quad (3.23)$$

From (3.22) and (3.23), we have

$$|\widehat{v}_j(\frac{n}{L_j}, 1)| \geq \frac{c}{4L_j} \quad (3.24)$$

on  $\Gamma_j^c$  defined by

$$\Gamma_j^c := \{1 \leq |n| \leq C_0 L_j\} \setminus \Gamma_j.$$

Now, let us estimate the size of  $\Gamma_j$ . From (3.15) and (3.23), we have

$$(\#\Gamma_j)^{\frac{1}{2}} \cdot \frac{c}{4L_j} \leq \left( \sum_{n \in \Gamma_j} \left| \widehat{w}_j(\frac{n}{L_j}, 1) - \widehat{v}_j(\frac{n}{L_j}, 1) \right|^2 \right)^{\frac{1}{2}} \leq CL_j^{-\frac{1}{2}} \delta_j \quad (3.25)$$

for sufficiently small  $\delta_j > 0$ . Thus, we have

$$\#\Gamma_j \lesssim L_j \delta_j^2. \quad (3.26)$$

Hence, from (2.3), (3.24), (3.26), and a Riemann sum approximation, we have

$$\begin{aligned} \|v_j(1)\|_{\dot{H}^s(\mathbb{T}_{L_j})} &\gtrsim L_j^{\frac{1}{2}} \left( \sum_{n \in \Gamma_j^c} \left| \frac{n}{L_j} \right|^{2s} \frac{1}{L_j^2} \right)^{\frac{1}{2}} \geq L_j^{\frac{1}{2}} \left( \sum_{L_j \delta_j^2 \lesssim |n| \leq C_0 L_j} \left| \frac{n}{L_j} \right|^{2s} \frac{1}{L_j^2} \right)^{\frac{1}{2}} \\ &\sim \left( \int_{c_0 \delta_j^2 \leq |\xi| \leq C_0} |\xi|^{2s} d\xi \right)^{\frac{1}{2}} \sim \delta_j^{2s+1} \end{aligned} \quad (3.27)$$

for all  $L_j = \frac{\delta_j}{\lambda_j} \gg 1$  and  $\delta_j \ll 1$ .

From (3.17), (3.14), and (3.27) with (3.21), we conclude that

$$\begin{aligned} \|u_j(\lambda_j^2)\|_{H^s(\mathbb{T})} &= \lambda_j^{-1} \left( \sum_{n \in \mathbb{Z}} (1 + |n|^2)^s |\widehat{v}_j(\frac{n}{L_j}, 1)|^2 \right)^{\frac{1}{2}} \\ &\geq \lambda_j^{-1} L_j^{s-\frac{1}{2}} \cdot L_j^{\frac{1}{2}} \left( \sum_{n \neq 0} \left( \frac{1}{L_j^2} + \left| \frac{n}{L_j} \right|^2 \right)^s |\widehat{v}_j(\frac{n}{L_j}, 1)|^2 \right)^{\frac{1}{2}} \\ &\sim \lambda_j^{-s-\frac{1}{2}} \delta_j^{s-\frac{1}{2}} \|v_j(1)\|_{\dot{H}^s(\mathbb{T}_{L_j})} \\ &\gtrsim \delta_j^\theta \cdot \delta_j^{2s+1} \gg j \end{aligned} \quad (3.28)$$

by choosing  $\delta_j > 0$  sufficiently small. Clearly, we can choose  $\lambda_j > 0$  sufficiently small to guarantee that  $t = \lambda_j^2 < \frac{1}{j}$ . Hence, (3.12) follows from (3.20) and (3.28).

**3.3. Critical case.** Next, we consider the critical case  $s = s_{\text{crit}} = -\frac{1}{2}$ . In this case, the scaling argument used in the previous subsection does not work any longer and the high-to-low energy transfer used in the supercritical case (as in [14] and Subsection 3.2 presented above) is not enough. We need to exploit a stronger high-to-low energy transfer for some solutions to (3.1). See Lemma 3.2 below.

Given  $N \gg 1$ , define a periodic function  $\phi_N$  on  $\mathbb{T}$  by setting

$$\widehat{\phi}_N(n) = \mathbf{1}_{N+Q_A}(n) + \mathbf{1}_{2N+Q_A}(n), \quad (3.29)$$

where  $Q_A = [-\frac{A}{2}, \frac{A}{2}]$  with

$$A = A(N) = \frac{N}{(\log N)^{\frac{1}{16}}}. \quad (3.30)$$

Let  $w^N$  be the global solution to (3.1) posed on  $\mathbb{T}$  such that  $w^N|_{t=0} = \phi_N$ . In view of (3.2), we have

$$w^N(t) = \phi_N e^{i|\phi_N|^2 t}. \quad (3.31)$$

By writing (3.1) in the integral form:

$$w^N(t) = \phi_N + i \int_0^t |w^N|^2 w^N(t') dt', \quad (3.32)$$

it follows from the algebra property of the Wiener algebra  $\mathcal{FL}^1(\mathbb{T})$  that we can also construct the solution  $w^N$  to (3.32) by a fixed point argument on a time interval  $[0, T_N^*]$  with

$$T_N^* \sim \|\phi_N\|_{\mathcal{FL}^1(\mathbb{T})}^{-2} \sim \frac{(\log N)^{\frac{1}{8}}}{N^2}, \quad (3.33)$$

satisfying

$$\|w^N(t)\|_{\mathcal{FL}^1(\mathbb{T})} \lesssim \|\phi_N\|_{\mathcal{FL}^1(\mathbb{T})} \sim \frac{N}{(\log N)^{\frac{1}{16}}} \quad (3.34)$$

for all  $t \in [0, T_N^*]$ . Moreover, by a variant of the persistence of regularity (with the triangle inequality and Young's inequality on the Fourier side), we have

$$\|w^N(t)\|_{\mathcal{FL}^\infty(\mathbb{T})} \lesssim \|\phi_N\|_{\mathcal{FL}^\infty(\mathbb{T})} = 1, \quad (3.35)$$

for all  $t \in [0, T_N^*]$ .

From (3.29) and (3.30), we have

$$\|\phi_N\|_{H^{-\frac{1}{2}}(\mathbb{T})} \sim (\log N)^{-\frac{1}{32}}, \quad (3.36)$$

which tends to 0 as  $N \rightarrow \infty$ . The following lemma exhibits a crucial norm inflation for the ODE (3.1) in the critical regularity.

**Lemma 3.2.** *Given  $N \gg 1$ , define  $T_N > 0$  by*

$$T_N = \frac{1}{N^2(\log N)^{\frac{1}{8}}}. \quad (3.37)$$

*Then, we have*

$$\|\mathbf{P}_{<N} w^N(T_N)\|_{H^{-\frac{1}{2}}(\mathbb{T})} \gtrsim (\log N)^{\frac{1}{4}}.$$

*Here,  $\mathbf{P}_{<N}$  denotes the projection onto the frequencies  $\{|n| < N\}$ .*

Lemma 3.2 follows from expressing (3.31) in the power series:

$$w^N(t) = \sum_{k=0}^{\infty} \Xi_k(t) := \sum_{k=0}^{\infty} \frac{(it)^k}{k!} |\phi_N|^{2k} \phi_N$$

and estimating each term, either from below or above. The main contribution comes from  $\Xi_1$ . We present the proof of Lemma 3.2 in Appendix B.

Let  $u^N$  be the solution to (1.1) with  $u^N|_{t=0} = \phi_N$ . We denote by  $\mathbf{u}^N$  the interaction representation of  $u^N$  defined by  $\mathbf{u}^N(t) = e^{it\partial_x^2} u^N(t)$ . Then, from

the Duhamel formulation of  $u^N$ ,<sup>5</sup> we have

$$\begin{aligned} \mathbf{u}^N(t) = \phi_N + i \int_0^t \sum_{n \in \mathbb{Z}} e^{inx} \\ \times \sum_{\Gamma(n)} e^{-i\Phi(\bar{n})t'} \widehat{\mathbf{u}}^N(n_1, t') \overline{\widehat{\mathbf{u}}^N(n_2, t')} \widehat{\mathbf{u}}^N(n_3, t') dt', \end{aligned} \quad (3.38)$$

where  $\Phi(\bar{n}) = n^2 - n_1^2 + n_2^2 - n_3^2$  and

$$\Gamma(n) = \{(n_1, n_2, n_3) \in \mathbb{Z}^3 : n = n_1 - n_2 + n_3\}.$$

Then, arguing as above for the solution  $w^N$  to (3.32), it follows from (3.38), the algebra property of  $\mathcal{FL}^1(\mathbb{T})$ , and a variant of the persistence of regularity that

$$\|\mathbf{u}^N(t)\|_{\mathcal{FL}^1(\mathbb{T})} \lesssim \|\phi_N\|_{\mathcal{FL}^1(\mathbb{T})}, \quad (3.39)$$

$$\|\mathbf{u}^N(t)\|_{\mathcal{FL}^\infty(\mathbb{T})} \lesssim \|\phi_N\|_{\mathcal{FL}^\infty(\mathbb{T})}, \quad (3.40)$$

for all  $t \in [0, T_N^*]$ , where  $T_N^*$  is as in (3.33). In the following, we use  $w^N$  to approximate the interaction representation  $\mathbf{u}^N$ .

**Lemma 3.3.** *Given  $N \in \mathbb{N}$ , let  $w^N$  and  $\mathbf{u}^N$  be as above. Then, for any  $T \in (0, T_N^*]$ , we have*

$$\begin{aligned} \|w^N - \mathbf{u}^N\|_{L_T^\infty \mathcal{FL}^\infty} \\ \lesssim \left\| \sum_{n=n_1-n_2+n_3} \int_0^t (1 - e^{-i\Phi(\bar{n})t'}) dt' \widehat{\phi}_N(n_1) \overline{\widehat{\phi}_N(n_2)} \widehat{\phi}_N(n_3) \right\|_{L_T^\infty \mathcal{FL}^\infty} \\ + \sum_{k=2}^4 T^k \|\phi_N\|_{\mathcal{FL}^1(\mathbb{T})}^{2k} \|\phi_N\|_{\mathcal{FL}^\infty(\mathbb{T})}, \end{aligned}$$

where the implicit constant is independent of  $N \in \mathbb{N}$  and  $L_T^\infty \mathcal{FL}^\infty := L^\infty([0, T]; \mathcal{FL}^\infty(\mathbb{T}))$ .

We first present the proof of Theorem 1.1, assuming Lemma 3.3. Given  $j \in \mathbb{N}$ , fix  $N = N(j) \gg 1$  such that

$$\frac{1}{N^2(\log N)^{\frac{1}{8}}} \ll \frac{1}{(\log N)^{\frac{1}{32}}} \ll \frac{1}{j} \quad \text{and} \quad (\log N)^{\frac{1}{4}} \gg j. \quad (3.41)$$

Let  $T_N$  as in (3.37). By the mean value theorem, we have

$$\sup_{t \in [0, T_N]} \left| \int_0^t (1 - e^{-i\Phi(\bar{n})t'}) dt' \right| \leq T_N^2 |\Phi(\bar{n})| \lesssim T_N^2 N^2, \quad (3.42)$$

---

<sup>5</sup>Henceforth, we drop the factor  $2\pi$  when it plays no role.

provided that  $|n|, |n_j| \lesssim N$ ,  $j = 1, 2, 3$ . Then, by Lemma 3.3 and (3.42) with (3.34) and (3.35), we have

$$\begin{aligned} & \|w^N - \mathbf{u}^N\|_{L_{T_N}^\infty \mathcal{FL}^\infty} \\ & \lesssim T_N^2 N^2 \|\phi_N\|_{\mathcal{FL}^1(\mathbb{T})}^2 \|\phi_N\|_{\mathcal{FL}^\infty(\mathbb{T})} + \sum_{k=2}^4 T_N^k \|\phi_N\|_{\mathcal{FL}^1(\mathbb{T})}^{2k} \|\phi_N\|_{\mathcal{FL}^\infty(\mathbb{T})} \\ & \lesssim (\log N)^{-\frac{3}{8}}. \end{aligned}$$

In particular, we have

$$\begin{aligned} & \|\mathbf{P}_{<N}(w^N(T_N) - \mathbf{u}^N(T_N))\|_{H^{-\frac{1}{2}}(\mathbb{T})} \\ & \lesssim (\log N)^{\frac{1}{2}} \|w^N(T_N) - \mathbf{u}^N(T_N)\|_{\mathcal{FL}^\infty(\mathbb{T})} \lesssim (\log N)^{\frac{1}{8}}. \end{aligned} \quad (3.43)$$

Then, from Lemma 3.2 and (3.43), we conclude that

$$\begin{aligned} & \|u^N(T_N)\|_{H^{-\frac{1}{2}}(\mathbb{T})} = \|\mathbf{u}^N(T_N)\|_{H^{-\frac{1}{2}}(\mathbb{T})} \\ & \geq \|\mathbf{P}_{<N}\mathbf{u}^N(T_N)\|_{H^{-\frac{1}{2}}(\mathbb{T})} \gtrsim (\log N)^{\frac{1}{4}}. \end{aligned} \quad (3.44)$$

Therefore, it follows from (3.36) and (3.44) with (3.37) and (3.41) that the desired estimates in (3.12) hold with  $t_j := T_N \ll \frac{1}{j}$ .

It remains to present the proof of Lemma 3.3.

*Proof of Lemma 3.3.* Note that Lemma 3.3 immediately follows once we prove

$$\begin{aligned} & \left\| \mathbf{u}^N - \phi_N - i \int_0^t e^{i\partial_x^2 t'} (|e^{-i\partial_x^2 t'} \phi_N|^2 e^{-i\partial_x^2 t'} \phi_N) dt' \right\|_{L_T^\infty \mathcal{FL}^\infty(\mathbb{T})} \\ & \lesssim \sum_{k=2}^4 T^k \|\phi_N\|_{\mathcal{FL}^1(\mathbb{T})}^{2k} \|\phi_N\|_{\mathcal{FL}^\infty(\mathbb{T})} \end{aligned} \quad (3.45)$$

and

$$\begin{aligned} & \left\| w^N - \phi_N - i \int_0^t |\phi_N|^2 \phi_N dt' \right\|_{L_T^\infty \mathcal{FL}^\infty(\mathbb{T})} \\ & \lesssim \sum_{k=2}^4 T^k \|\phi_N\|_{\mathcal{FL}^1(\mathbb{T})}^{2k} \|\phi_N\|_{\mathcal{FL}^\infty(\mathbb{T})} \end{aligned} \quad (3.46)$$

for  $0 \leq T \leq T_N^*$ . We only prove (3.45) since (3.46) follows in a similar manner. In the following, we drop the superscripts and subscripts  $N$  for simplicity. Moreover, we set  $\phi_n = \widehat{\phi}(n)$ ,  $\mathbf{u}_n(t) = \widehat{\mathbf{u}}(n, t)$ , and so on.

In view of the (3.38), we have

$$\mathbf{u}_n(t) = \phi_n + i \int_0^t \sum_{\Gamma(n)} e^{-i\Phi(\bar{n})t'} \mathbf{u}_{n_1} \bar{\mathbf{u}}_{n_2} \mathbf{u}_{n_3} dt' =: \phi_n + \mathcal{N}(\mathbf{u})_n. \quad (3.47)$$

By substituting  $\mathbf{u} = \phi + \mathcal{N}(\mathbf{u})$  in (3.47), we obtain

$$\begin{aligned} \mathbf{u}_n(t) &= \phi_n + i \int_0^t \sum_{\Gamma(n)} e^{-i\Phi(\bar{n})t'} (\phi_{n_1} + \mathcal{N}(\mathbf{u})_{n_1}) \\ &\quad \times \overline{(\phi_{n_2} + \mathcal{N}(\mathbf{u})_{n_2})} (\phi_{n_3} + \mathcal{N}(\mathbf{u})_{n_3}) dt' \\ &= \phi_n + i \int_0^t \sum_{\Gamma(n)} e^{-i\Phi(\bar{n})t'} \phi_{n_1} \bar{\phi}_{n_2} \phi_{n_3} dt' + \text{Error}_n, \end{aligned}$$

where the terms in  $\text{Error}_n$  consist of higher order terms of the form:

$$i \int_0^t \sum_{\Gamma(n)} e^{-i\Phi(\bar{n})t'} v_{n_1} \bar{v}_{n_2} v_{n_3} dt' \quad (3.48)$$

with  $v = \phi$  or  $\mathcal{N}(\mathbf{u})$ , not all three factors equal to  $\phi$ . Then, it suffices to show that

$$\sup_{t \in [0, T]} \sup_n |\text{Error}_n| \lesssim \sum_{k=2}^4 T^k \|\phi\|_{\mathcal{F}L^1(\mathbb{T})}^{2k} \|\phi\|_{\mathcal{F}L^\infty(\mathbb{T})}. \quad (3.49)$$

Note that (3.49) follows from (3.48), (3.47), and Young's inequality with (3.39) and (3.40). The proof of (3.46) follows in a similar manner with (3.34) and (3.35).  $\square$

This completes the proof of Theorem 1.1 in the critical case.

#### 4. ON NORM INFLATION FOR THE CUBIC FRACTIONAL NLS

In this section, we present the proof of Theorem 1.2 for the fractional NLS (1.6).

**4.1. Supercritical case.** In this case, the following cubic fractional NLS with small dispersion:

$$i\partial_t v + \delta^{2\alpha}(-\partial_x^2)^\alpha v + |v|^2 v = 0 \quad (4.1)$$

plays an important role. We state a variant of Lemma 3.1.

**Lemma 4.1.** *Given  $\phi \in C_c^\infty(\mathbb{R})$ , suppose that  $L$  is sufficiently large such that  $\text{supp } \phi \subset \mathbb{T}_L$ . Let  $w$  and  $v = v(\delta)$  be the solutions to (3.1) and (4.1) on  $\mathbb{T}_L$  with  $w|_{t=0} = v|_{t=0} = \phi$ , respectively. Then, given any  $\beta \in (0, 2\alpha)$ , there exist  $c > 0$ ,  $C > 0$ ,  $\delta_0 > 0$ , and  $L_0 \geq 1$  such that*

$$\|v(t) - w(t)\|_{H^1(\mathbb{T}_L)} \leq C\delta^\beta$$

for all  $|t| \leq c|\log \delta|^c$ , all  $0 < \delta \leq \delta_0$ , and all periods  $L \geq L_0$ .

The proof of Lemma 4.1 is basically the same as that of Lemma 3.1 and hence we omit details.

We divide the supercritical case  $s < s_{\text{crit}}^\alpha = \frac{1}{2} - \alpha$  into the following three cases:

$$(i) \ s \leq -\frac{1}{2}, \quad (ii) \ -\frac{1}{2} < s < 0, \quad \text{and} \quad (iii) \ s > 0.$$



Recall that there is no inflation when  $s = 0$  due to the  $L^2$ -conservation. When  $s \leq -\frac{1}{2}$ , the basic structure of the proof is the same as that of Theorem 1.1. When  $s > -\frac{1}{2}$ , however, we need to employ a more robust high-to-low energy transfer mechanism for the ODE (3.1) as in Subsection 3.3.

• **Case (i)**  $s \leq -\frac{1}{2}$ . In this case, we follow the argument presented in Subsection 3.2. Let  $w$  be the solution to (3.1) on  $\mathbb{R}$  with  $w|_{t=0} = \phi$  considered in Subsection 3.2. See Appendix A for the construction of such  $\phi$ . In particular,  $w$  satisfies (3.13). Given  $j \in \mathbb{N}$ , let  $0 < \lambda_j \ll \delta_j \ll 1$ . With  $L_j = \frac{\delta_j}{\lambda_j} \gg 1$  as in (3.14), let  $w_j$  and  $v_j$  be the solutions to (3.1) and (4.1) with  $\delta = \delta_j$  on  $\mathbb{T}_{L_j}$  such that  $w_j|_{t=0} = v_j|_{t=0} = \phi$ . By choosing  $\delta_j \leq \delta_0$ , Lemma 4.1 yields

$$\|v_j(t) - w_j(t)\|_{H^1(\mathbb{T}_{L_j})} \leq C\delta_j^\alpha \quad (4.2)$$

for all  $|t| \leq c|\log \delta_j|^c$ . Now, define  $u_j$  by

$$u_j(x, t) = \frac{1}{\lambda_j^\alpha} v_j\left(\frac{\delta_j x}{\lambda_j}, \frac{t}{\lambda_j^{2\alpha}}\right). \quad (4.3)$$

Then,  $u_j$  is a solution to (1.6) on  $\mathbb{T}$ . From (2.1), we have

$$\widehat{u}_j(n, \lambda_j^{2\alpha} t) = \frac{1}{\lambda_j^\alpha} \widehat{v}_j\left(\frac{n}{L_j}, t\right). \quad (4.4)$$

As before, we have

$$\widehat{u}_j(0, 0) = \frac{1}{\lambda_j^\alpha} \widehat{v}_j(0, 0) = \frac{1}{\lambda_j^\alpha L_j} \mathcal{F}_{\mathbb{R}}(\phi)(0) = 0. \quad (4.5)$$

From (2.3) and (4.4), we have

$$\|u_j(t)\|_{\dot{H}^s(\mathbb{T})} = \lambda_j^{-s+\frac{1}{2}-\alpha} \delta_j^{s-\frac{1}{2}} \|v_j\left(\frac{t}{\lambda_j^{2\alpha}}\right)\|_{\dot{H}^s(\mathbb{T}_{L_j})}. \quad (4.6)$$

In particular, in view of (4.5), (4.6), and (2.7) with  $v_j(0) = \phi$ , we can choose  $0 < \lambda_j \ll \delta_j \ll 1$  such that

$$\|u_j(0)\|_{\dot{H}^s(\mathbb{T})} \leq \|u_j(0)\|_{\dot{H}^s(\mathbb{T})} \sim \lambda_j^{-s+\frac{1}{2}-\alpha} \delta_j^{s-\frac{1}{2}} \|\phi\|_{\dot{H}^s(\mathbb{R})} \ll \frac{1}{j}, \quad (4.7)$$

since  $s < s_{\text{crit}}^\alpha = \frac{1}{2} - \alpha$ . In particular, we can choose  $0 < \lambda_j \ll \delta_j \ll 1$  such that

$$\lambda_j^{-s+\frac{1}{2}-\alpha} \delta_j^{s-\frac{1}{2}} \sim \begin{cases} \delta_j^\theta \ll \frac{1}{j}, & \text{if } s < -\frac{1}{2}, \\ \{\log(C_1 \delta_j^{-2\alpha})\}^{-\frac{1}{4}} \ll \frac{1}{j}, & \text{if } s = -\frac{1}{2}, \end{cases} \quad (4.8)$$

for some small  $\theta > 0$  with  $s < -\frac{1}{2} - \frac{\theta}{2\alpha}$  in the first case, where  $C_1 > 0$  is to be specified later.

Let  $\Gamma_j$  be as in (3.23). By repeating the computation in (3.25) with (4.2), we obtain

$$\#\Gamma_j \lesssim L_j \delta_j^{2\alpha}. \quad (4.9)$$

Using (4.9), we proceed as in (3.27), where the domain of the integration is replaced by  $\{c_0\delta_j^{2\alpha} \leq |\xi| \leq C_0\}$ . Then, setting  $C_1 := c_0^{-1}C_0$ , we have

$$\|v_j(1)\|_{\dot{H}^s(\mathbb{T}_{L_j})} \gtrsim \begin{cases} \delta_j^{\alpha(2s+1)} \gg 1, & \text{if } s < -\frac{1}{2}, \\ \{\log(C_1\delta_j^{-2\alpha})\}^{\frac{1}{2}} \gg 1, & \text{if } s = -\frac{1}{2}. \end{cases} \quad (4.10)$$

Finally, proceeding as in (3.28) with (4.4), (3.14), (4.8), and (4.10), we obtain

$$\begin{aligned} \|u_j(\lambda_j^{2\alpha})\|_{H^s(\mathbb{T})} &= \lambda_j^{-\alpha} \left( \sum_{n \in \mathbb{Z}} (1 + |n|^2)^s |\widehat{v}_j(\frac{n}{L_j}, 1)|^2 \right)^{\frac{1}{2}} \\ &\geq \lambda_j^{-\alpha} L_j^{s-\frac{1}{2}} \cdot L_j^{\frac{1}{2}} \left( \sum_{n \neq 0} \left( \frac{1}{L_j^2} + |\frac{n}{L_j}|^2 \right)^s |\widehat{v}_j(\frac{n}{L_j}, 1)|^2 \right)^{\frac{1}{2}} \\ &\sim \lambda_j^{-s+\frac{1}{2}-\alpha} \delta_j^{s-\frac{1}{2}} \|v(1)\|_{\dot{H}^s(\mathbb{T}_{L_j})} \\ &\gtrsim \begin{cases} \delta_j^\theta \cdot \delta_j^{\alpha(2s+1)} \gg j, & \text{if } s < -\frac{1}{2}, \\ \{\log(C_1\delta_j^{-2\alpha})\}^{\frac{1}{4}} \gg j, & \text{if } s = -\frac{1}{2}, \end{cases} \end{aligned} \quad (4.11)$$

by choosing  $\delta_j > 0$  sufficiently small. We can also choose  $\lambda_j > 0$  sufficiently small such that  $t_j = \lambda_j^{2\alpha} < \frac{1}{j}$ . Hence, Theorem 1.2 follows from (4.7) and (4.11) in this case.

• **Case (ii)**  $-\frac{1}{2} < s < 0$ . Note that the high-to-low energy transfer manifested in (3.13) is not sufficient to show a norm inflation when  $s > -\frac{1}{2}$ . Hence, we first need to construct another solution to the ODE (3.1), exhibiting a more robust high-to-low energy transfer.

Given  $N \gg 1$ , define a function  $\psi_N$  on  $\mathbb{R}$  by setting

$$\widehat{\psi}_N(\xi) = R\{\mathbf{1}_{N+Q_A}(\xi) + \mathbf{1}_{2N+Q_A}(\xi)\}, \quad (4.12)$$

where  $R = R(N)$  and  $A = A(N)$  are given by

$$R = R(N) = \frac{1}{N^s \log N} \quad \text{and} \quad A = A(N) = \log N.$$

From (4.12), we have

$$\|\psi_N\|_{H^s(\mathbb{R})} \sim (\log N)^{-\frac{1}{2}}. \quad (4.13)$$

We first state a norm inflation for the ODE (3.1) posed on  $\mathbb{R}$  when  $-\frac{1}{2} < s < 0$ .

**Lemma 4.2.** *Let  $-\frac{1}{2} < s < 0$ . Given  $N \gg 1$ , let  $w_N$  be the global solution to (3.1) posed on  $\mathbb{R}$  with  $w_N|_{t=0} = \psi_N$ . Then, we have*

$$\|w_N(T_N)\|_{H^s(\mathbb{R})} \gtrsim N^{-s} (\log N)^{-\frac{3}{2}+s},$$

where  $T_N$  is defined by

$$T_N = \frac{N^{2s}}{\log N}. \quad (4.14)$$

The proof of Lemma 4.2 is analogous to that of Lemma 3.2 and is presented in Appendix B.

Given large  $j \in \mathbb{N}$ , choose  $N = N(j) \gg 1$  such that

$$\frac{N^{2s}}{\log N} \ll \frac{1}{(\log N)^{\frac{1}{2}}} \ll \frac{1}{j} \quad \text{and} \quad N^{-s}(\log N)^{-\frac{3}{2}+s} \gg j. \quad (4.15)$$

Given  $s < 0$ , fix  $\kappa \in \mathbb{N}$  such that  $s + \kappa \geq 0$  and let  $\psi_N^\kappa = \partial_x^{-\kappa} \psi_N$ . More precisely,  $\psi_N^\kappa$  is defined by  $\widehat{\psi_N^\kappa}(\xi) = (2\pi i \xi)^{-\kappa} \widehat{\psi_N}(\xi)$ . Given small  $\eta > 0$  (to be chosen later), let  $\phi^\kappa \in C_c^\infty(\mathbb{R})$  such that

$$\|\phi^\kappa - \psi_N^\kappa\|_{H^{\kappa+1}(\mathbb{R})} < \eta. \quad (4.16)$$

Define  $\phi$  by setting  $\phi = \partial_x^\kappa \phi^\kappa$ . Then, from (4.16) have

$$\|\phi - \psi_N\|_{H^1(\mathbb{R})} \leq \|\phi^\kappa - \psi_N^\kappa\|_{H^{\kappa+1}(\mathbb{R})} < \eta. \quad (4.17)$$

Recalling that  $s + \kappa \geq 0$ , it follows from (4.16) and (4.12) that

$$\begin{aligned} \|\mathbf{P}_{\leq 1} \phi\|_{\dot{H}^s(\mathbb{R})} &= \|\mathbf{P}_{\leq 1} \phi^\kappa\|_{\dot{H}^{s+\kappa}(\mathbb{R})} \\ &\leq \|\phi^\kappa - \psi_N^\kappa\|_{H^{\kappa+1}(\mathbb{R})} + \|\mathbf{P}_{\leq 1} \psi_N^\kappa\|_{\dot{H}^{s+\kappa}(\mathbb{R})} < \eta. \end{aligned} \quad (4.18)$$

Hence, from (2.7), (4.13), (4.17), and (4.18) with (4.15), we have

$$\|\phi\|_{\dot{H}^s(\mathbb{T}_L)} \sim \|\phi\|_{\dot{H}^s(\mathbb{R})} \lesssim (\log N)^{-\frac{1}{2}} + \eta \ll \frac{1}{j}, \quad (4.19)$$

for all sufficiently large  $L \gg 1$ , provided that  $\eta \ll \frac{1}{j}$ .

Let  $w$  be the solution to (3.1) on  $\mathbb{R}$  with  $w|_{t=0} = \phi$ . Then, by the continuity of the solution map to (3.1) (in the  $H^1$ -topology) and (4.17), we can choose  $\eta = \eta(N) > 0$  sufficiently small to guarantee that

$$\|w(t) - w_N(t)\|_{H^1(\mathbb{R})} \lesssim 1 \quad (4.20)$$

for all  $t \in [0, T_N]$ . Then, from Lemma 4.2 with (4.15) and (4.20), we have

$$\|w(T_N)\|_{H^s(\mathbb{R})} \gg j. \quad (4.21)$$

Now, recalling that  $s < s_{\text{crit}} = \frac{1}{2} - \alpha$ , choose  $0 < \lambda_j \ll \delta_j \ll 1$  such that

$$\lambda_j^{-s+\frac{1}{2}-\alpha} \delta_j^{s-\frac{1}{2}} \sim 1 \quad (4.22)$$

and  $\text{supp } \phi \subset \mathbb{T}_{L_j}$ , where  $L_j = \frac{\delta_j}{\lambda_j} \gg 1$  is as in (3.14). Moreover, from (2.8) and (4.21), we have

$$\|w(T_N)\|_{H^s(\mathbb{T}_{L_j})} \gg j \quad (4.23)$$

by choosing  $L_j$  sufficiently large. Let  $v_j$  be the solution to (4.1) with  $\delta = \delta_j$  on  $\mathbb{T}_{L_j}$  such that  $v_j|_{t=0} = \phi$ . Then, it follows from Lemma 4.1 applied to  $w$  and  $v_j$  on  $[0, T_N]$  with (4.14), (4.15), and (4.23) that

$$\|v_j(T_N)\|_{H^s(\mathbb{T}_{L_j})} \gg j.$$

This in particular implies

$$\|v_j(T_N)\|_{\dot{H}^s(\mathbb{T}_{L_j})} \gg j \quad \text{or} \quad L_j^{\frac{1}{2}} |\widehat{v}_j(0, T_N)| \gg j. \quad (4.24)$$

Now, define  $u_j$  by (4.3). On the one hand, it follows from (4.5), (4.6), (4.19), and (4.22) that

$$\|u_j(0)\|_{H^s(\mathbb{T})} \leq \|u_j(0)\|_{\dot{H}^s(\mathbb{T})} = \lambda_j^{-s+\frac{1}{2}-\alpha} \delta_j^{s-\frac{1}{2}} \|\phi\|_{\dot{H}^s(\mathbb{T}_{L_j})} \ll \frac{1}{j}. \quad (4.25)$$

On the other hand, from (4.4), (4.6), (4.22), and (4.24), we obtain

$$\begin{aligned} \|u_j(\lambda_j^{2\alpha} T_N)\|_{H^s(\mathbb{T})} &\sim \|u_j(\lambda_j^{2\alpha} T_N)\|_{\dot{H}^s(\mathbb{T})} + |\widehat{u}_j(0, \lambda_j^{2\alpha} T_N)| \\ &= \lambda_j^{-s+\frac{1}{2}-\alpha} \delta_j^{s-\frac{1}{2}} \|v_j(T_N)\|_{\dot{H}^s(\mathbb{T}_{L_j})} + \frac{1}{\lambda_j^\alpha} |\widehat{v}_j(0, T_N)| \\ &\sim \|v_j(T_N)\|_{\dot{H}^s(\mathbb{T}_{L_j})} + L_j^{\frac{1}{2}-s} |\widehat{v}_j(0, T_N)| \\ &\gg j. \end{aligned} \quad (4.26)$$

Note that we have  $t_j = \lambda_j^{2\alpha} T_N \ll \frac{1}{j}$  in view of (4.14) and (4.15). Hence, Theorem 1.2 follows from (4.25) and (4.26) in this case.

• **Case (iii)** Lastly, we consider the case  $0 < s < \frac{1}{2} - \alpha$ . Fix a non-constant mean-zero function  $\phi$  with a compact support on  $\mathbb{R}$  and let  $w$  be the solution to (3.1) on  $\mathbb{R}$  with  $w|_{t=0} = \phi$ . From a direct calculation, we have

$$\|w(t)\|_{\dot{H}^k(\mathbb{T}_L)} = \|w(t)\|_{\dot{H}^k(\mathbb{R})} \sim (1+t)^k.$$

for all  $k \in \mathbb{N} \cup \{0\}$  and all sufficiently large  $L \gg 1$ . Then, by interpolation, we have

$$\|w(t)\|_{\dot{H}^\sigma(\mathbb{T}_L)} \sim (1+t)^\sigma \quad (4.27)$$

for all  $\sigma \geq 0$ .

Given  $j \in \mathbb{N}$ , choose  $0 < \lambda_j \ll \delta_j \ll 1$  such that (4.27) holds for  $\sigma = s$  and  $L = L_j := \frac{\delta_j}{\lambda_j} \gg 1$ . Let  $v_j$  be the solution to (4.1) with  $\delta = \delta_j$  on  $\mathbb{T}_{L_j}$  such that  $v_j|_{t=0} = \phi$ . Then, by Lemma 4.1 and (4.27), there exists  $c_0 > 0$  such that

$$\|v_j(t)\|_{\dot{H}^s(\mathbb{T}_{L_j})} \sim (1+t)^s. \quad (4.28)$$

for  $|t| \leq c_0 |\log \delta_j|^{c_0}$ , provided  $\delta_j \ll 1$  and  $L_j \gg 1$ . In the following, we further impose that

$$\lambda_j^{-s+\frac{1}{2}-\alpha} \delta_j^{s-\frac{1}{2}} \sim |\log \delta_j|^{-\frac{1}{2}c_0 s}. \quad (4.29)$$

Then, it follows from (4.5), (4.6), (4.29), and (2.7) that

$$\|u_j(0)\|_{H^s(\mathbb{T})} \lesssim \|u_j(0)\|_{\dot{H}^s(\mathbb{T})} \sim |\log \delta_j|^{-\frac{1}{2}c_0s} \|\phi\|_{\dot{H}^s(\mathbb{R})} \ll \frac{1}{j} \quad (4.30)$$

for sufficiently small  $\delta_j > 0$  and  $L_j \gg 1$ . Letting  $\tilde{t}_j = c_0 |\log \delta_j|^{c_0}$ , it follows from (4.6), (4.28), (4.29) that

$$\|u_j(\lambda_j^{2\alpha} \tilde{t}_j)\|_{H^s(\mathbb{T})} \geq \|u_j(\lambda_j^{2\alpha} \tilde{t}_j)\|_{\dot{H}^s(\mathbb{T})} \sim |\log \delta_j|^{\frac{1}{2}c_0s} \gg j \quad (4.31)$$

for sufficiently small  $\delta_j > 0$ . From (4.29), we have  $\lambda_j \lesssim \delta_j^\beta$  for some  $\beta > 0$ . Hence, by choosing  $\delta_j > 0$  sufficiently small, we have

$$t_j := \lambda_j^{2\alpha} \tilde{t}_j \lesssim \delta_j^{2\alpha\beta} \cdot c_0 |\log \delta_j|^{c_0} \ll \frac{1}{j}. \quad (4.32)$$

Therefore, Theorem 1.2 follows from (4.30), (4.31), and (4.32) in this case.

**4.2. Critical case.** Next, we consider the critical case  $s = s_{\text{crit}}^\alpha = \frac{1}{2} - \alpha \leq -\frac{1}{2}$ . In view of Theorem 1.1, we assume that  $s < -\frac{1}{2}$ . The argument follows closely the presentation in Subsection 3.3 with Lemma 4.3 below replacing Lemma 3.2.

Given  $N \gg 1$ , define a periodic function  $\phi_N$  on  $\mathbb{T}$  by setting

$$\widehat{\phi}_N(n) = R\{\mathbf{1}_{N+Q_A}(n) + \mathbf{1}_{2N+Q_A}(n)\}, \quad (4.33)$$

where  $R = R(N)$  and  $A = A(N)$  are given by

$$R = R(N) = N^{-\frac{1}{2}-s} \quad \text{and} \quad A = A(N) = N^{1-\theta}, \quad (4.34)$$

with sufficiently small  $\theta > 0$  such that

$$s < -\frac{1}{2} - 3\theta.$$

From (4.33) and (4.34), we have

$$\|\phi_N\|_{H^{\frac{1}{2}-\alpha}(\mathbb{T})} \sim N^{-\frac{\theta}{2}}. \quad (4.35)$$

Let  $w^N$  be the global solution to (3.1) posed on  $\mathbb{T}$  such that  $w^N|_{t=0} = \phi_N$ . Proceeding as in Subsection 3.3, we have

$$\|w^N(t)\|_{\mathcal{FL}^1(\mathbb{T})} \lesssim \|\phi_N\|_{\mathcal{FL}^1(\mathbb{T})} \sim N^{\frac{1}{2}-s-\theta}, \quad (4.36)$$

$$\|w^N(t)\|_{\mathcal{FL}^\infty(\mathbb{T})} \lesssim \|\phi_N\|_{\mathcal{FL}^\infty(\mathbb{T})} \sim N^{-\frac{1}{2}-s} \quad (4.37)$$

for all  $t \in [0, T_N^*]$ , where  $T_N^*$  is given by

$$T_N^* \sim \|\phi_N\|_{\mathcal{FL}^1(\mathbb{T})}^{-2} \sim N^{2s-1+2\theta}. \quad (4.38)$$

The following lemma is a variant of Lemma 3.2 for  $s < -\frac{1}{2}$ . See Appendix B for the proof.

**Lemma 4.3.** *Let  $s < -\frac{1}{2}$ . Given  $N \gg 1$ , define  $T_N > 0$  by*

$$T_N = N^{2s-1-\theta}. \quad (4.39)$$

*Then, we have*

$$\|\mathbf{P}_{<N} w^N(T_N)\|_{H^{\frac{1}{2}-\alpha}(\mathbb{T})} \gtrsim N^{-\frac{1}{2}-s-3\theta}.$$

Let  $u^N$  be the solution to (1.6) with  $u^N|_{t=0} = \phi_N$ . Then, let  $\mathbf{u}^N$  be the interaction representation of  $u^N$  defined by  $\mathbf{u}^N(t) = e^{it(-\partial_x^2)^\alpha} u^N(t)$ . From the Duhamel formulation of  $u^N$ , we have

$$\begin{aligned} \mathbf{u}^N(t) &= \phi_N + i \int_0^t \sum_{n \in \mathbb{Z}} e^{inx} \\ &\quad \times \sum_{\Gamma(n)} e^{-i\Phi_\alpha(\bar{n})t'} \widehat{\mathbf{u}^N}(n_1, t') \overline{\widehat{\mathbf{u}^N}(n_2, t')} \widehat{\mathbf{u}^N}(n_3, t') dt', \end{aligned} \quad (4.40)$$

where  $\Phi_\alpha(\bar{n}) = |n|^{2\alpha} - |n_1|^{2\alpha} + |n_2|^{2\alpha} - |n_3|^{2\alpha}$ . Then, as in Subsection 3.3, we have (3.39) and (3.40) for all  $t \in [0, T_N^*]$ , where  $T_N^*$  is as in (4.38). Moreover, the approximation lemma (Lemma 3.3) also holds with  $w^N$ ,  $\mathbf{u}^N$ , and  $T_N^*$  in our current context.

Given  $j \in \mathbb{N}$ , fix  $N = N(j) \gg 1$  such that

$$N^{2s-1-\theta} \ll N^{-\frac{\theta}{2}} \ll \frac{1}{j} \quad \text{and} \quad N^{-\frac{1}{2}-s-3\theta} \gg j. \quad (4.41)$$

Let  $T_N = N^{-2\alpha-\theta}$  as in (4.39). Then, by the mean value theorem, we have

$$\sup_{t \in [0, T_N]} \left| \int_0^t (1 - e^{-i\Phi(\bar{n})t'}) dt' \right| \leq T_N^2 |\Phi_\alpha(\bar{n})| \lesssim T_N^2 N^{2\alpha}, \quad (4.42)$$

provided that  $|n|, |n_j| \lesssim N$ ,  $j = 1, 2, 3$ . Then, by Lemma 3.3 and (4.42) with (4.36) and (4.37), we have

$$\begin{aligned} &\|w^N - \mathbf{u}^N\|_{L_{T_N}^\infty \mathcal{FL}^\infty} \\ &\lesssim T_N^2 N^{2\alpha} \|\phi_N\|_{\mathcal{FL}^1(\mathbb{T})}^2 \|\phi_N\|_{\mathcal{FL}^\infty(\mathbb{T})} + \sum_{k=2}^4 T_N^k \|\phi_N\|_{\mathcal{FL}^1(\mathbb{T})}^{2k} \|\phi_N\|_{\mathcal{FL}^\infty(\mathbb{T})} \\ &\lesssim N^{-\frac{1}{2}-s-4\theta}. \end{aligned} \quad (4.43)$$

In particular, we have

$$\begin{aligned} \|\mathbf{P}_{<N}(w^N(T_N) - \mathbf{u}^N(T_N))\|_{H^s(\mathbb{T})} &\lesssim \|w^N(T_N) - \mathbf{u}^N(T_N)\|_{\mathcal{FL}^\infty(\mathbb{T})} \\ &\lesssim N^{-\frac{1}{2}-s-4\theta}. \end{aligned} \quad (4.44)$$

Then, from Lemma 4.3 and (4.44), we conclude that

$$\begin{aligned} \|u^N(T_N)\|_{H^s(\mathbb{T})} &= \|\mathbf{u}^N(T_N)\|_{H^s(\mathbb{T})} \\ &\geq \|\mathbf{P}_{<N} \mathbf{u}^N(T_N)\|_{H^s(\mathbb{T})} \gtrsim N^{-\frac{1}{2}-s-3\theta}. \end{aligned} \quad (4.45)$$

Therefore, Theorem 1.2 in the critical case follows from (4.35) and (4.45) with (4.39) and (4.41).

**Remark 4.4.** It follows from the proof of Lemma 4.3 (see Lemma B.4 below) that

$$\|\mathbf{P}_{<N} w^N(T_N)\|_{H^s(\mathbb{T})} \gtrsim T_N \|\phi_N\|_{\mathcal{F}L^1(\mathbb{T})}^2 \|\phi_N\|_{\mathcal{F}L^\infty(\mathbb{T})} \sim N^{-\frac{1}{2}-s-3\theta} \quad (4.46)$$

and  $T_N N^{2\alpha} \sim N^{-\theta}$ . Comparing (4.43) and (4.46), we see that it is essential to have  $T_N N^{2\alpha} \lesssim 1$  for our argument to work.

**Remark 4.5.** The proof of Theorem 1.2 in the critical case with  $s < -\frac{1}{2}$  is based on Lemma 4.3, exploiting a high-to-low energy transfer analogous to Lemmas 3.2 and 4.2 along with the approximation lemma (Lemma 3.3). When  $-\frac{1}{2} < s < 0$ , one may hope to exploit a similar high-to-low energy transfer. There are, however, no possible values of  $R$  and  $A$  for an initial condition  $\phi_N$  of the form (4.33), guaranteeing norm inflation at time  $T_N \lesssim N^{-2\alpha}$ . See Remark B.5.

#### APPENDIX A. HIGH-TO-LOW ENERGY TRANSFER FOR THE ODE: PART 1

Let  $s \leq -\frac{1}{2}$ . In the following, we discuss the construction of the function  $\phi \in C_c^\infty(\mathbb{R}) \cap \dot{H}^s(\mathbb{R})$  described in Subsection 3.2, satisfying (i)  $\widehat{\phi}(\xi) = O_{\xi \rightarrow 0}(|\xi|^\kappa)$  for some  $\kappa > -s - \frac{1}{2}$  and (ii) there exist  $t_0 > 0$  and  $c > 0$  such that

$$\left| \int_{\mathbb{R}} w(x, t_0) dx \right| \geq c, \quad (A.1)$$

where  $w$  is the solution (3.1) on  $\mathbb{R}$  with  $w|_{t=0} = \phi$ . In view of (3.2), we can rewrite (A.1) as

$$\left| \int_{\mathbb{R}} \phi(x) e^{i|\phi(x)|^2 t_0} dx \right| \geq c. \quad (A.2)$$

The main addition from [14] is the compactness of the support of  $\phi$ , which is crucial for our application in the periodic setting.

The following lemma states that the smoothness assumption on  $\phi$  is inessential.

**Lemma A.1.** *Let  $s \leq -\frac{1}{2}$ . Suppose that there exists a function  $\psi \in C_c^0(\mathbb{R}) \cap \dot{H}^s(\mathbb{R})$  such that (i)  $\widehat{\psi}(\xi) = O_{\xi \rightarrow 0}(|\xi|^\kappa)$  for some  $\kappa > -s - \frac{1}{2}$  and (ii) there exist  $t_0 > 0$  and  $c > 0$  such that*

$$\left| \int_{\mathbb{R}} \psi(x) e^{i|\psi(x)|^2 t_0} dx \right| \geq c. \quad (A.3)$$

*Then, there exists  $\phi \in C_c^\infty(\mathbb{R}) \cap \dot{H}^s(\mathbb{R})$  satisfying both the conditions (i) and (ii) with  $c$  replaced by  $\frac{c}{2}$ .*

*Proof.* It follows from Paley-Wiener Theorem [34, Theorem IX.12] that  $\widehat{\psi}$  is an analytic function on  $\mathbb{R}$ . Hence, the vanishing of  $\widehat{\psi}$  at 0 must occur at an integral order. This allows us to assume that  $\kappa \in \mathbb{N}$ . Then, from the Taylor expansion  $\widehat{\psi}(\xi) = \sum_{j=0}^{\infty} \frac{\partial^j \widehat{\psi}(0)}{j!} \xi^j$ , we have  $\partial^j \widehat{\psi}(0) = 0$  for  $j = 0, 1, \dots, \kappa - 1$ . In other words, we have

$$\int_{\mathbb{R}} x^j \psi(x) dx = 0 \quad (\text{A.4})$$

for  $j = 0, 1, \dots, \kappa - 1$ .

Let  $\eta \in C_c^\infty(\mathbb{R})$  be a smooth bump function with  $\text{supp } \eta \subset [-1, 1]$  and  $\int \eta dx = 1$ . Given  $\varepsilon > 0$  (to be chosen later), let  $\phi_\varepsilon = \eta_\varepsilon * \psi$ , where  $\eta_\varepsilon(x) = \varepsilon^{-1} \eta(\varepsilon^{-1}x)$ . Then, we claim that

$$\int_{\mathbb{R}} x^j \phi_\varepsilon(x) dx = 0 \quad (\text{A.5})$$

for  $j = 0, 1, \dots, \kappa - 1$ . Given  $j \in \{0, 1, \dots, \kappa - 1\}$ , write  $x^j$  as

$$x^j = ((x - y) + y)^j = \sum_{k=0}^j c_k(y) (x - y)^k.$$

Then, by Fubini's theorem, we have

$$\int_{\mathbb{R}} x^j \phi_\varepsilon(x) dx = \sum_{k=0}^j \int \left( \int (x - y)^k \psi(x - y) dx \right) c_k(y) \eta_\varepsilon(y) dy = 0.$$

Hence, (A.5) holds. Noting  $\phi_\varepsilon$  has a compact support, we conclude from the analyticity of  $\widehat{\phi}_\varepsilon$  that the condition (i) also holds for  $\phi_\varepsilon$ .

For  $0 < \varepsilon \leq 1$ , we have  $\text{supp } \phi_\varepsilon \subset \text{supp } \psi + B_0(1)$ , where  $B_0(1)$  denotes the ball of radius 1 centered at 0. In particular,  $\phi_\varepsilon$  converges to  $\psi$  in  $L^\infty(\mathbb{R})$ . Then, by the triangle inequality and the mean value theorem, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}} \psi(x) e^{i|\psi(x)|^2 t_0} dx - \int_{\mathbb{R}} \phi_\varepsilon(x) e^{i|\phi_\varepsilon(x)|^2 t_0} dx \right| \\ & \leq \int_{\mathbb{R}} |\psi(x)| |e^{i|\psi(x)|^2 t_0} - e^{i|\phi_\varepsilon(x)|^2 t_0}| dx + \int_{\mathbb{R}} |\psi(x) - \phi_\varepsilon(x)| dx \\ & \leq c(\psi, t_0) \int_{\text{supp } \psi + B_0(1)} |\psi(x) - \phi_\varepsilon(x)| dx \leq \frac{c}{2} \end{aligned} \quad (\text{A.6})$$

for all sufficiently small  $\varepsilon > 0$ . Therefore, from (A.3) and (A.6), we conclude that the condition (ii) with  $c$  replaced by  $\frac{c}{2}$  also holds for  $\phi_\varepsilon$  with sufficiently small  $\varepsilon > 0$ .  $\square$

For the first few values of  $\kappa \in \mathbb{N}$ , we concretely construct functions  $\psi_\kappa$ , satisfying the conditions (i) and (ii) in Lemma A.1. Then, Lemma A.1 yields a smoothed version  $\phi_\kappa$  of  $\psi_\kappa$ , serving as a good initial condition  $\phi$  in the supercritical case of the proof of Theorem 1.1 and 1.2 when  $-\kappa - \frac{1}{2} < s \leq -\frac{1}{2}$ .



Define  $\psi_1$  by

$$\psi_1(x) = \sqrt{\pi}\mathbf{1}_{[1,3]}(x) - 2\sqrt{\pi}\mathbf{1}_{[4,5]}(x).$$

Note that  $\widehat{\psi}_1(0) = 0$ . Thus, by the analyticity of  $\widehat{\psi}_1$ , we have  $\widehat{\psi}_1(\xi) = O_{\xi \rightarrow 0}(|\xi|)$ . The condition (ii) is clearly satisfied with  $t_0 = 1$ .

Now, let  $\psi_2(x) = \psi_1(x) + \psi_1(-x)$ . It is easy to see that  $\psi_2$  satisfies (A.4) for  $j = 0, 1$ . Thus, by analyticity of  $\widehat{\psi}_2$ , it satisfies the condition (i) with  $\kappa = 2$ . The condition (ii) is clearly satisfied with  $t_0 = 1$ .

Next, define  $\psi_4$  with a parameter  $a$  by

$$\psi_4(x; a) = \sqrt{\pi}\mathbf{1}_{[1,2]}(x) - 2\sqrt{\pi}\mathbf{1}_{[4,5]}(x) + \sqrt{\pi}\mathbf{1}_{[a, a+1]}(x), \quad x \geq 0$$

and  $\psi_4(x; a) = \psi_4(-x; a)$  for  $x < 0$ , where  $a > 5$ . By definition,  $\psi_4$  satisfies (A.4) for  $j = 0, 1$ , and 3. Letting  $F(a) = \int_{\mathbb{R}} x^2 \psi_4(x; a) dx$ , we see that  $F(5) < 0 < F(10)$ . Hence, by the intermediate value theorem, there exists  $a_* \in (5, 10)$  such that  $F(a_*) = 0$ . Now, set  $\psi_4(x) = \psi_4(x; a_*)$ . Then, it is easy to see that  $\psi_4$  satisfies the conditions (i) and (ii)<sup>6</sup> with  $\kappa = 4$ .

In general, one may continue to construct  $\psi_\kappa$  in this fashion (and construct smooth  $\phi_\kappa$  by applying Lemma A.1) but combinatorics gets cumbersome. In the following, we instead discuss an alternative (simpler but less direct) construction of  $\phi_\kappa \in C_c^\infty(\mathbb{R}) \cap \dot{H}^s(\mathbb{R})$ ,  $\kappa \in \mathbb{N}$ , satisfying the conditions (i) and (ii).

Let  $w$  be a smooth solution to (3.1) with a smooth initial condition  $w|_{t=0} = \phi$ . Then, from (3.1), we have

$$\partial_t^2 \int_{\mathbb{R}} w dx = - \int_{\mathbb{R}} |w|^4 w dx.$$

In particular, by taking a mean-zero real-valued smooth initial condition  $\phi \not\equiv 0$  with a compact support such that  $\int \phi^5 dx < 0$ , we have

$$\partial_t^2 \int_{\mathbb{R}} w dx \Big|_{t=0} = - \int_{\mathbb{R}} \phi^5 dx > 0.$$

Then, by continuity in time of  $w$ , we have

$$\partial_t^2 \operatorname{Re} \int_{\mathbb{R}} w(t) dx = - \operatorname{Re} \int_{\mathbb{R}} |w|^4 w(t) dx > 0 \quad (\text{A.7})$$

on some time interval  $[0, t_0]$ . On the other hand, from (3.1), we have

$$\partial_t \operatorname{Re} \int_{\mathbb{R}} w dx \Big|_{t=0} = \operatorname{Re} \left( i \int_{\mathbb{R}} \phi^3 dx \right) = 0. \quad (\text{A.8})$$

Hence, it follows from (A.7) and (A.8) that

$$\partial_t \operatorname{Re} \int_{\mathbb{R}} w(t) dx > 0$$

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<sup>6</sup>We can write  $\psi_4 = \sqrt{\pi}\mathbf{1}_{E_1} - 2\sqrt{\pi}\mathbf{1}_{E_2}$ , where  $E_1 \cap E_2 = \emptyset$ . It is clear that functions of this form satisfy (A.3) at  $t_0 = 1$  with  $c = \sqrt{\pi}|E_1| + 2\sqrt{\pi}|E_2| > 0$ .

for  $t \in (0, t_0]$ . Recalling that  $w|_{t=0} = \phi$  has mean 0, we obtain

$$\operatorname{Re} \int_{\mathbb{R}} w(t) dx > 0$$

for  $t \in (0, t_0]$ . This implies (A.1).

Given  $\kappa \in \mathbb{N}$ , let  $f$  be a real-valued smooth function on  $\mathbb{R}$  with a compact support such that  $\int (\partial_x^\kappa f)^5 dx < 0$ . Then, it is easy to see that  $\phi_\kappa := \partial_x^\kappa f \in C_c^\infty(\mathbb{R})$  satisfies the conditions (i) and (ii).

## APPENDIX B. HIGH-TO-LOW ENERGY TRANSFER FOR THE ODE: PART 2

In this appendix, we construct solutions to the ODE (3.1) with more robust high-to-low energy transfer than those constructed in Appendix A.

Let  $\mathcal{M} = \mathbb{T}$  or  $\mathbb{R}$  and  $\widehat{\mathcal{M}}$  denote the Pontryagin dual of  $\mathcal{M}$  given by  $\widehat{\mathcal{M}} = \mathbb{Z}$  if  $\mathcal{M} = \mathbb{T}$  and  $\widehat{\mathcal{M}} = \mathbb{R}$  if  $\mathcal{M} = \mathbb{R}$ . Let  $s < 0$ . Given  $N \gg 1$ , define a function  $\phi_N$  on  $\mathcal{M}$  by

$$\widehat{\phi}_N(\xi) = R\{\mathbf{1}_{N+Q_A}(\xi) + \mathbf{1}_{2N+Q_A}(\xi)\}, \quad \xi \in \widehat{\mathcal{M}}, \quad (\text{B.1})$$

for suitably chosen  $A = A(N) \gg 1$  and  $R = R(N) \geq 1$ , where  $Q_A = [-\frac{A}{2}, \frac{A}{2}]$ . Note that we have

$$\|\phi_N\|_{H^s(\mathcal{M})} \sim RA^{\frac{1}{2}} N^s. \quad (\text{B.2})$$

Let  $w = w(N)$  be the solution to (3.1) with  $w|_{t=0} = \phi_N$ . Then, Lemmas 3.2, 4.2, and 4.3 follow as a corollary to the following proposition.

**Proposition B.1.** *Given  $N \gg 1$ , set  $s < 0$ ,  $R = R(N)$ ,  $A = A(N)$ , and  $T_N > 0$  by one of the followings:*

$$(i) \ s = -\frac{1}{2}, \quad R = 1, \quad A = \frac{N}{(\log N)^{\frac{1}{16}}} \quad \text{and} \quad T_N = \frac{1}{N^2(\log N)^{\frac{1}{8}}}, \quad (\text{B.3})$$

$$(ii) \ s < 0, \quad R = \frac{1}{N^s \log N}, \quad A = \log N, \quad \text{and} \quad T_N = \frac{N^{2s}}{\log N}, \quad (\text{B.4})$$

or

$$(iii) \ s < -\frac{1}{2}, \quad R = N^{-\frac{1}{2}-s}, \quad A = N^{1-\theta}, \quad \text{and} \quad T = N^{2s-1-\theta}, \quad (\text{B.5})$$

where  $\theta > 0$  is sufficiently small such that

$$s < -\frac{1}{2} - 3\theta. \quad (\text{B.6})$$

Then, we have

$$\begin{aligned} \|w(T_N)\|_{H^s(\mathcal{M})} &\geq \|\mathbf{P}_{<N} w(T_N)\|_{H^s(\mathcal{M})} \\ &\gtrsim \begin{cases} (\log N)^{\frac{1}{4}}, & \text{if (i) holds,} \\ N^{-s}(\log N)^{-2}g(N), & \text{if (ii) holds,} \\ N^{-\frac{1}{2}-s-3\theta}, & \text{if (iii) holds,} \end{cases} \end{aligned} \quad (\text{B.7})$$

where  $g(N)$  is given by

$$g(N) = \begin{cases} 1, & \text{if } s < -\frac{1}{2}, \\ (\log \log N)^{\frac{1}{2}}, & \text{if } s = -\frac{1}{2}, \\ (\log N)^{\frac{1}{2}+s}, & \text{if } -\frac{1}{2} < s < 0. \end{cases}$$

Note that, when (ii) holds, we only use Proposition B.1 for  $-\frac{1}{2} < s < 0$ . We nonetheless include the proof for all  $s < 0$  in the following.

**Remark B.2.** From (B.2), we have

$$\|\phi_N\|_{H^s(\mathcal{M})} \sim \begin{cases} (\log N)^{-\frac{1}{32}}, & \text{if (i) holds,} \\ (\log N)^{-\frac{1}{2}}, & \text{if (ii) holds,} \\ N^{-\frac{\theta}{2}}, & \text{if (iii) holds,} \end{cases} \quad (\text{B.8})$$

all tending to 0 as  $N \rightarrow \infty$ .

From the explicit formula (3.2) and the power series expansion, we have

$$w(t) = \phi_N e^{i|\phi_N|^2 t} = \sum_{k=0}^{\infty} \Xi_k(t), \quad (\text{B.9})$$

where  $\Xi_k$  is defined by

$$\Xi_k(t) := \frac{(it)^k}{k!} |\phi_N|^{2k} \phi_N. \quad (\text{B.10})$$

We prove Proposition B.1 by estimating each  $\Xi_k$  either from above or below. We first state elementary lemmas.

**Lemma B.3.** *Let  $s < 0$ . Then, there exists  $C > 0$  such that*

$$\|\Xi_k(t)\|_{H^s(\mathcal{M})} \leq \frac{C^k t^k}{k!} (RA)^{2k} R \cdot f(A) \quad (\text{B.11})$$

for any  $k \in \mathbb{N}$ , where  $f(A)$  is given by

$$f(A) = \begin{cases} 1, & \text{if } s < -\frac{1}{2}, \\ (\log A)^{\frac{1}{2}}, & \text{if } s = -\frac{1}{2}, \\ A^{\frac{1}{2}+s}, & \text{if } s > -\frac{1}{2}. \end{cases} \quad (\text{B.12})$$

*Proof.* From (B.1), we see that  $\text{supp } \widehat{\phi}_N$  consists of two disjoint intervals of length  $A$ . Since  $\Xi_k(t)$  is basically a  $(2k+1)$ -fold product of  $\phi_N$  and its complex conjugate, it follows that the spatial support of  $\mathcal{F}[\Xi_k(t)]$  consists of (at most)  $2^{2k+1}$  intervals of length  $A$ . Thus, we have

$$|\text{supp } \mathcal{F}[\Xi_k(t)]| \leq C^k A.$$

for some  $C > 0$ . By the monotonicity of  $\langle \xi \rangle^s$  for  $s < 0$ , we have

$$\|\langle \xi \rangle^s\|_{L^2_\xi(\text{supp } \mathcal{F}[\Xi_k(t)])} \leq \|\langle \xi \rangle^s\|_{L^2_\xi([- \frac{1}{2} C^k A, \frac{1}{2} C^k A])} \lesssim C^k f(A). \quad (\text{B.13})$$

Then, by Hölder's inequality, (B.13), and Young's inequality with (B.1), we have

$$\begin{aligned} \|\Xi_k(t)\|_{H^s(\mathcal{M})} &\leq \|\langle \xi \rangle^s\|_{L_\xi^2(\text{supp } \mathcal{F}[\Xi_k(t)])} \|\Xi_k(t)\|_{\mathcal{FL}^\infty} \\ &\leq f(A) \cdot \frac{C^k t^k}{k!} \|\phi_N\|_{\mathcal{FL}^1}^{2k} \|\phi_N\|_{\mathcal{FL}^\infty} \\ &\leq \frac{C^k t^k}{k!} (RA)^{2k} R \cdot f(A). \end{aligned}$$

This proves (B.11).  $\square$

In the next lemma, we show that Lemma B.3 is indeed sharp when  $k = 1$ , by exploiting a high-to-low energy transfer mechanism in  $\Xi_1$ .

**Lemma B.4.** *Let  $s < 0$  and  $A \ll N$ . Then, we have*

$$\|\Xi_1(t)\|_{H^s} \geq \|\mathbf{P}_{<N} \Xi_1(t)\|_{H^s} \gtrsim t R^3 A^2 \cdot f(A), \quad (\text{B.14})$$

where  $f(A)$  is as in (B.12).

*Proof.* First, recall the following simple lemma on the convolution of characteristic functions of intervals:

$$\mathbf{1}_{a+Q_A} * \mathbf{1}_{b+Q_A}(\xi) \gtrsim A \cdot \mathbf{1}_{a+b+Q_A}(\xi) \quad (\text{B.15})$$

for all  $a, b, \xi \in \widehat{\mathcal{M}}$  and  $A \geq 1$ . Then, from (B.10), (B.1), and (B.15), we have

$$|\mathcal{F}[\Xi_1(t)](\xi)| = t |\widehat{\phi}_N * \widehat{\phi_N} * \widehat{\phi_N}(\xi)| \gtrsim t R^3 A^2 \cdot \mathbf{1}_{Q_A}(\xi).$$

Then, (B.14) follows once we note that  $\|\langle \xi \rangle^s\|_{L_\xi^2(Q_A)} \sim f(A)$  and  $A \ll N$ .  $\square$

We now present the proof of Proposition B.1.

*Proof of Proposition B.1.* From (B.3), (B.4), and (B.5), we have

$$T_N R^2 A^2 = \begin{cases} (\log N)^{-\frac{1}{4}} \ll 1, & \text{if (i) holds,} \\ (\log N)^{-1} \ll 1, & \text{if (ii) holds,} \\ N^{-3\theta} \ll 1, & \text{if (iii) holds.} \end{cases}$$

Then, from (B.9) with (B.8) and Lemma B.3, we have

$$\begin{aligned} \|w(T_N) - \Xi_1(T_N)\|_{H^s} &\leq \|\Xi_0(T_N)\|_{H^s} + \left\| \sum_{k=2}^{\infty} \Xi_k(T_N) \right\|_{H^s} \\ &\lesssim \begin{cases} 1, & \text{if (i) holds,} \\ N^{-s} (\log N)^{-3} g(N), & \text{if (ii) holds,} \\ N^{-\frac{\theta}{2}} + N^{-\frac{1}{2}-s-6\theta}, & \text{if (iii) holds.} \end{cases} \quad (\text{B.16}) \end{aligned}$$

On the other hand, from Lemma B.4, we have

$$\begin{aligned} \|\Xi_1(T_N)\|_{H^s} &\geq \|\mathbf{P}_{<N}\Xi_1(T_N)\|_{H^s} \\ &\gtrsim \begin{cases} (\log N)^{\frac{1}{4}}, & \text{if (i) holds,} \\ N^{-s}(\log N)^{-2}g(N), & \text{if (ii) holds,} \\ N^{-\frac{1}{2}-s-3\theta}, & \text{if (iii) holds.} \end{cases} \end{aligned} \quad (\text{B.17})$$

Therefore, the desired estimate (B.7) follows from (B.16) and (B.17) with (B.6).  $\square$

**Remark B.5.** Let us briefly discuss the situation for Theorem 1.2 in the critical case when  $s = s_{\text{crit}}^\alpha = \frac{1}{2} - \alpha \in (-\frac{1}{2}, 0)$ . In order to prove an analogue of Proposition B.1 for  $-\frac{1}{2} < s < 0$ , the following must hold:

- (a)  $RA^{\frac{1}{2}}N^s =: D \ll 1$ ,
- (b)  $TR^2A^2 \ll 1$ ,
- (c)  $TR^3A^{\frac{5}{2}+s} \gg 1$ ,

with  $A = A(N) \ll N$  and  $D = D(N) \rightarrow 0$  as  $N \rightarrow \infty$ . Moreover, in carrying out the argument in Subsection 4.2, we also need to have

- (d)  $T \lesssim N^{-2\alpha}$ .

Let  $A = EN$  and  $T = FN^{2s-1}$  for some  $E \ll 1$  and  $F \lesssim 1$ . Then, from (a), (b), and (d), we have

$$TR^2A^2 = EF \cdot R^2AN^{2s} = D^2EF.$$

Then, from (c), we obtain

$$1 \ll TR^3A^{\frac{5}{2}+s} = E^s \cdot TR^2A^2 \cdot RA^{\frac{1}{2}}N^s = D^3E^{1+s}F.$$

Hence, we must have

$$D^{-3}E^{-1-s} \ll F \lesssim 1.$$

This is clearly a contradiction since  $D \rightarrow 0$ ,  $E \ll 1$  and  $-\frac{1}{2} < s < 0$ . Therefore, even if norm inflation at the critical regularity holds true in this case, one needs to develop a new method to prove it.

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